The Borel complexity of the spaces of left-orderings

joint with A. Clay (U Manitoba)

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1. Left-orderable groups

Let G be a countable group.

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G is **bi-orderable** if it admits a strict total order such that for all $f, g, h \in G$,

$$g < h \quad \Longrightarrow \quad (fg < fh \quad \text{and} \quad gf < hf).$$

Examples of left-orderable groups

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- Torsion-free Abelian groups are bi-orderable.

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Counterexample

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- Finite index subgroups of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (Witte-Morris '94).
- Irreducible lattices in a real semi-simple Lie group with finite center and real rank at least two. (Deroin–Hurtado '20+)

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Suppose that K and H are left-orderable and that

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 1,$$

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Then for any pair of left-orders $<_K$ and $<_H$ we can define a left-order on G by

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Example

The group $G = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ is left-orderable as witnessed by

$$1 \longrightarrow \langle a \rangle \xrightarrow{i} G \xrightarrow{q} G/\langle a \rangle \longrightarrow 1.$$

However G is not bi-orderable.

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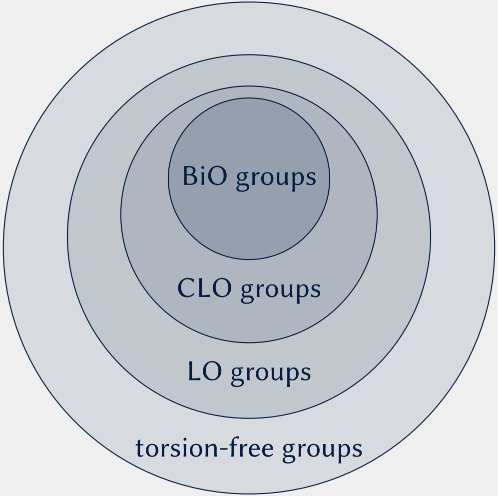
Proposition

The following are equivalent:

- *G admits a Conradian left-order;*
- *G is **locally indicable**, i.e., for every finitely generated $H \leq G$ there is an onto homomorphism $H \to \mathbb{Z}$.*

Refining torsion-freeness



2. The space of left-orders

Proposition

G is **left-orderable** iff there is $P \subseteq G$ such that

1. P is a subsemigroup of G;
2. $G = P \sqcup P^{-1} \sqcup \{1_G\}$.

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If $<$ is a left-order on G then the **positive cone** $P_< = \{g \in G \mid 1_G < g\}$ satisfies 1 and 2.

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If $<$ is a left-order on G then the **positive cone** $P_< = \{g \in G \mid 1_G < g\}$ satisfies 1 and 2. Conversely, if $P \subseteq G$ satisfies 1 and 2, then define a left-order on G by

$$g <_P h \iff g^{-1}h \in P.$$

□

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Definition (Sikora '04; Ghys)

The **space of left-orders** on G can be defined as

$$\mathrm{LO}(G) := \{P \subseteq G \mid P \text{ satisfies (1) and (2)}\}$$

- We regard $\mathrm{LO}(G) \subseteq 2^G = \{x \mid x\colon G \to \{0,1\}\}$ with the subspace topology.

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- We regard $\mathrm{LO}(G) \subseteq 2^G = \{x \mid x\colon G \to \{0,1\}\}$ with the subspace topology.
- Since $\mathrm{LO}(G)$ is closed, it is a **compact Polish subspace** of 2^G.

The continuous **conjugacy action** $G \curvearrowright \mathrm{LO}(G)$ is defined by

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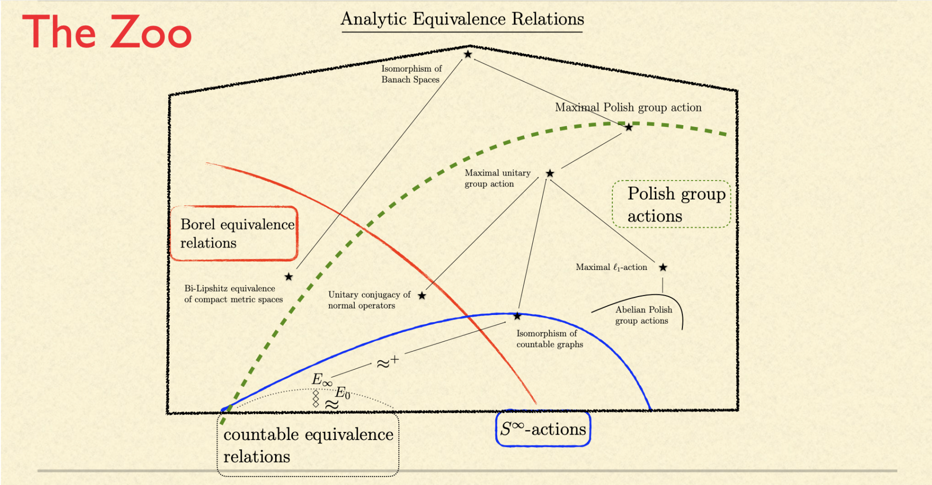
$E_{\mathsf{lo}}(G)$ is a **countable Borel equivalence relation (cber)**.



Courtesy of Matt Foreman

Question (Deroin, Navas, Rivas, *Groups Orders Dynamics*, 2016)

Is there any left-orderable group G such that $\mathrm{LO}(G)/E_{\mathsf{lo}}(G)$ is not a standard Borel space?

A QUESTION FROM *GOD*

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Equivalently,

*Is there any left-orderable group G such that $E_{\mathsf{lo}}(G)$ is **not smooth**?*

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"Groups, Orders, and Dynamics" (2016)

Recall that E on X is **smooth** iff there is some Polish space Y and a Borel map $\varphi\colon X \to Y$ such that

$$x \mathrel{E} y \iff \varphi(x) = \varphi(y).$$

Descriptive necessity — local vs. global

The spaces $\mathrm{LO}(\mathbb{Z}^2)$ and $\mathrm{LO}(\mathbb{F}_2)$ are locally the same.

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Theorem (Sikora 04; Navas '10)

The spaces $\mathrm{LO}(\mathbb{Z}^2)$ *and* $\mathrm{LO}(\mathbb{F}_2)$ *have no isolated points.*

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Theorem (Sikora 04; Navas '10)

The spaces $\mathrm{LO}(\mathbb{Z}^2)$ and $\mathrm{LO}(\mathbb{F}_2)$ have no isolated points.

Only global properties of those spaces will show the extent to which they differ from each other.

3. Some results about Borel complexity

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- If $a \in A$ and $P \in \mathrm{LO}(G)$, then $a^{-1}Pa = P$.
- Let $\{g_1, \ldots, g_n\}$ be a set of left coset representatives for A in G. It follows that $G \cdot P = \{g_1^{-1}Pg_1, \ldots, g_n^{-1}Pg_n\}$. □

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Question

Does the converse hold?

Consequences of generic ergodicity

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Proposition

If E_Γ^X is smooth, then

1. *There must be $x_0 \in X$ with $\Gamma \cdot x_0$ finite.*
2. *There is a subgroup $N \leq \Gamma_{x_0} := \{g \in \Gamma \mid g \cdot x_0 = x_0\}$ such that*

$$N \lhd \Gamma \text{ and } [\Gamma : N] < \infty.$$

Corollary (C.–Clay 2022)

If G is not locally indicable then $E_{\mathsf{lo}}(G)$ is not smooth.

Proof.

If $E_{\mathsf{lo}}(G)$ is smooth then there is $P \in \mathrm{LO}(G)$ such that $G \cdot P$ is finite. Let $g \in P$ (i.e., $1_G <_P g$).

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Non-standard

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Therefore P is the positive cone of a Conradian left-order. □

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Question

Is there any locally indicable simple group that is not bi-orderable?

Corollary (C.–Clay '22)

If G is not locally indicable then $E_{\mathsf{lo}}(G)$ is not smooth.

Corollary (C.–Clay '23+)

If G has no non-trivial finite quotient (e.g., G is simple) and not bi-orderable, then $E_{\mathsf{lo}}(G)$ is not smooth.

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...but, fortunately, we can find many examples of groups that are locally indicable, not bi-orderable, and have no non-trivial finite quotients.

Lemma

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Lemma

If G is bi-orderable, then for any $g, h \in G$ and $m, n \in \mathbb{Z}$ we have

$$[g^m, h^n] = 1 \implies [g, h] = 1$$

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Example

It is well-known that the commutator subgroup F' of Thompson's group F is **simple** and **locally indicable** (in fact, it is bi-orderable).

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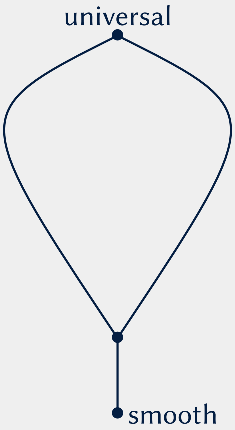
$E_{\mathsf{lo}}(K)$ is not smooth.

4. Time permitting...

Definition

E is **universal** if every cber is Borel reducible to E.



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There is a unique universal cber up to Borel reducibility, denoted by E_∞.

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Theorem (C.–Clay 2022)

$E_{\mathsf{lo}}(\mathbb{F}_2)$ *is a universal countable Borel equivalence relation.*

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It uses the following:

Lemma (C.–Clay 2022)

If $C \leq G$ and C is convex in some left-ordering of G and

$$\textit{for all } g \in G \quad gCg^{-1} \subseteq C \implies g \in C,$$

then $E_{\mathrm{lo}}(C) \leq_B E_{\mathrm{lo}}(G)$.

Theorem (C.–Clay 2023+)

If M is (the ***complement of a****) knot (excluding the trivial knot), then $E_{\mathsf{lo}}(\pi_1(M))$ is* ***not smooth****.*

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*If M is (the **complement of a**) knot (excluding the trivial knot), then $E_{\mathsf{lo}}(\pi_1(M))$ is **not smooth**.*

We build a nonempty invariant closed subset of $\mathrm{LO}(\pi_1(M))$ consisting of non-Conradian left-orderings (whose orbits are necessarily infinite).

Open question

Is there a left-orderable G such that $E_0 \leq_B E_{\mathsf{lo}}(G) \leq_B E_\infty$.

Open questions

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Open question

Are there G, H such that $E_{\mathsf{lo}}(G)$ and $E_{\mathsf{lo}}(H)$ are incomparable.

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Are there G, H such that $E_{\mathsf{lo}}(G)$ and $E_{\mathsf{lo}}(H)$ are incomparable.

Open question

Is there G such that $E_{\mathsf{lo}}(G)$ is essentially free?

Thank You!